Inverse Matrices

Definition

Two $n \times n$ matrices A and B are said to be **inverses** of one another iff

$$AB = BA = I_n \qquad (\text{ we write } B = A^{-1})$$

Recall

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For example

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 5 & -4 \end{bmatrix}$$

(A and B are **not** inverses of one another)

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{C}{\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(A and C are inverses of one another)

How to find inverses

Let A be an $n \times n$ matrix. Augment A with the $n \times n$ identity matrix and transform the augmented matrix to its *RREF*. If the *RREF* of A is I_n, A^{-1} exists (see diagram below). Otherwise, the matrix A has no inverse (**singular**).

$$\left[\; A \;\middle|\; I_n \;\right] \xrightarrow{RREF} \left[\; I_n \;\middle|\; A^{-1} \;\right]$$

Problem 1:

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

Therefore, $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$.

Problem 2:

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

and use it to solve the system

$$\begin{aligned} x + 3y + 3z &= 1 \\ x + 4y + 3z &= 2 \\ x + 3y + 4z &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Recall

If $A\vec{x} = \vec{b}$ and A^{-1} exists, then $\vec{x} = A^{-1}\vec{b}$.

Therefore, if we write the system in the matrix equation form

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ then } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix}.$$

<u>Problem 3</u>

Find the inverse of $B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \to$$ The matrix B has no inverse (**singular**).

Theorem 2

- The inverse of a matrix is **unique**.

  Proof

  Assume an $n \times n$ matrix A has two different inverses, B and C. Then $AB = BA = I_n$ and $AC = CA = I_n$. Then $B = I_n B = CAB = C(AB) = CI_n = C$.

- $\left(A^{-1}\right)^{-1} = A$

  Proof

  $AA^{-1} = I_n$

- $\left(AB\right)^{-1} = B^{-1}A^{-1}$

  Proof

  $AB\,B^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$

Theorem 3 (invertible matrices)

Let A be an $n \times n$ matrix. The following statements are **equivalent**.

1. A is invertible
2. The reduced row echelon form of A is I_n
3. The system $A\vec{x} = \vec{b}$ has a unique solution for any $n \times 1$ vector $\vec{b}$ $\left(\vec{x} = A^{-1}\vec{b}\right)$
4. The system $A\vec{x} = \vec{0}$ has only the trivial solution $\left(\vec{x} = \vec{0}\right)$
5. The columns of A span $\mathbb{R}^n$
6. The columns of A are independent
7. The columns of A form a basis for $\mathbb{R}^n$
8. The dimension of the null space of A is zero
9. $\det A \neq 0$
10. The matrix A does not have a zero eigenvalue

Homework

1. Find the inverse of

a. $A = \begin{bmatrix} 1 & 1 \\ 15 & 20 \end{bmatrix}$ b. $B = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

2. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

3. Show that $(cA)^{-1} = \frac{1}{c}A^{-1}$ (c : constant).

4. Show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{bc}{a(ad-bc)} & \frac{-b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \to$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$